Application of Combinatorics to Decide the Most Optimal Move in the Liar's Dice Game

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***Abstract*— This paper explores the application of combinatorics to determine the most optimal move in the Liar's Dice game. By using mathematical principles such as permutations, combinations, and probability theory. A probability based approach is proposed to evaluate optimal strategies based on revealed and hidden dice values. This research provides a structured mathematical foundation to enhance decision-making skills in the game, balancing deception and strategy without relying on advanced AI techniques. The insights presented aim to deepen understanding of probability-driven decision-making in dice-based games.**

***Keywords*—Liar, Dice, Combinatoric, Optimal.**

I. INTRODUCTION

A dice is one of the most popular object used in tabletop games. This small genius object, is a 6 sided cube with each side containing a dot representing a number between 1 to 6. Players usually roll this object on the table and use the top side of the dice as a number they use for certain actions. For example, in the game monopoly, which I'm sure we all at least heard about before, player use the sum of 2 randomized number from rolling 2 dice for moving around the board.

Liar's Dice, a game which originates from the 15th century, is a popular game for two or more players that uses a lot of dice. This tricky game requires the skill to deceive and detect an opponent's deception.

Image 1. Liar's dice game. Source : https://www.wikihow.com/Play-Liar%27s-Dice

The game's rules are quite simple. Each player will get 5 dice and a cup used for concealment. Then, every player roll their set of dice without revealing them to the other players. After the dice are rolled, the first player will have to make a bidding. A bid consist of any face value of a dice and the frequency the player believes that the dice are showing that face value under all of the players cup. Ones dice are considered a wild number which means it is considered to be every value all at once.

Each player then take turns in a circular order to either counter the previous player's bid with a higher one or challenge the previous bid, typically by calling "liar" hence the game's name. There are many variants on what is a allowed and disallowed for counter bidding, The one this paper use is the variant where the player may bid a higher quantity of any particular face or the same quantity of a higher face. If a challenge is made, all of the dice are revealed and counted. If the challenger is correct, the challenger may throw away a dice in their hand. But, if the bidder is correct, the bidder is the one who throw away a dice in their hand. The first person who loses all of their dice first wins the game.

There are also other version called the "Common Hand" for only two players (it technically can be player for more players, but it will be harder to play). The initial game is the same, each player roll their 5 dices and make sure to not reveal their values to the opponent. The difference is each "bid" is now similar to poker hands. For example, there are runt (any combinations), pair, two pairs, low straight (value from 1 to 5), three of a kind, full house (a pair and three of a kind of different value), high straight (value from 2 to 6), four of a kind, and five of a kind.

Category	Example
Five of a kind	⚃ ⚃ ⚃ ⚃ ⚃
Four of a kind	⚁ ⚁ ⚁ ⚁ ⚄
High straight	⚁ ⚂ ⚃ ⚄ ⚅
Full house	⚅ ⚅ ⚀ ⚀ ⚀
Three of a kind	⚃ ⚃ ⚃ ⚂ ⚁
Low straight	⚀ ⚁ ⚂ ⚃ ⚄
Two pair	⚁ ⚁ ⚄ ⚄ ⚀
Pair	⚅ ⚅ ⚄ ⚂ ⚁
Runt	⚀ ⚂ ⚃ ⚄ ⚅

Image 2. Bid example in common hand version. Source: https://en.wikipedia.org/wiki/Liar%27s_dice

The first person will call what their 5 set of dice is either lying or telling the truth. Then, the opponent will either challenge it or make a higher hand ranking call. Before making a higher hand ranking call, the player could re-roll any amount of their dice and announce the amount truthfully. When a challenge is made, if the call was correct, the challenger loses a point, if it was bluff and it was not right, the caller loses a point. The player who lost all of their points first loses the game.

II. Combinatorics Background

Combinatorics is a branch of mathematics that studies the art of enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties [1]. It plays a vital role in different areas like understanding how different configurations of elements can arise to analyzing their properties under specific rules.

At the most basic principles there are two rules: sum rules and product rules. Sum rule is used when you want to count the number of ways any of two events happening. For example, event A can be done in **m** ways and event B can be done in **n** ways, then the total of ways either event A or event B happens is **m** + **n**. Meanwhile, the product rule is used when you want to count the number of ways both events happening. We use events A and events B again for the example, then the total ways of both event A and event B happens is **m** * **n**.

Advancing even deeper from the basic principles, combinatorics mainly focuses on two main concepts: permutations and combinations. Permutations focuses on the arrangement of objects where the sequence is significant. While combinations focuses on the selecting group of objects without considering the order of the sequence.

The number of permutations of a set containing n elements is n! (n factorial), which is the product of integers from 1 to n. This calculation accounts for every possible ordering of the elements. For example, a set with three elements has 3! = 3 × 2 × 1 = 6 permutations.

When looking at permutations of a set containing r elements from a larger set of n elements, the number of possible permutations is:

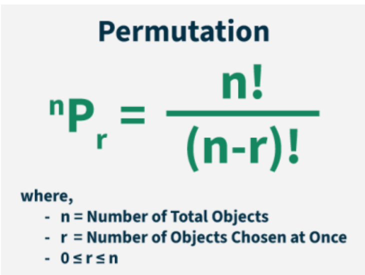

Image 3. Permutations formula. Source: https://www.geeksforgeeks.org/permutation/

This formula calculates the number of ways to arrange r elements selected from n elements, with considering the order of the elements.

The number of combinations, on the other hand, containing r elements from a set of n elements is:

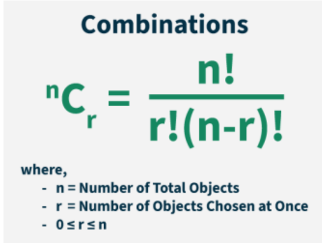

Image 4. Combinations formula. Source: https://www.geeksforgeeks.org/combinations/

This formula calculates the number of ways to arrange r elements selected from n elements, without considering the order of the elements.

One of the most interesting use cases of combinatorics is probability. By counting all of the possible outcomes, we can calculate probabilities in scenarios of randomization that ranges using dice to cards. By also using probabilities, we can optimize problems like determining the most efficient ways to do something.

Probability can be defined as numerical likehood of occurrence of an event. The probability of an event taking place lie between 0 and 1. The probability formula is as follows:

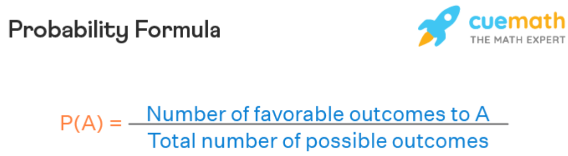

Image 5. Probability formula. Source: https://www.cuemath.com/data/probability-theory/

III. Implementation

A. Original Version

The original version uses 5 * number of players amount of dice as the starting hand. Note that for every bid a player does, on every single player's perspective, the chance for that bid is correct is different because of their respective revealed 5 dice. For an extreme example, in a two player game, player A has 5 dice of sixes and player B has 5 dice of ones. In the player B perspective, when player A bid 5 sixes, the probability of that happening is extremely low since player A know that it has 0 sixes. Meanwhile, from the player B perspective, the probability of that bid is actually 1 or 100%, since it knows that it has 5 sixes already. So, to calculate this, the probability of any non-one face value bid with quantity q when the player in perspective already has h amount dice with faces one or the chosen face value and n amount of unknown dice values (the opponents total dice) is:

$$Probability(q, h) = \sum_{i=q-h}^{n} C(n, i) \cdot \left(\frac{2}{6}\right)^{i} \cdot \left(\frac{4}{6}\right)^{n-i}$$

Also note that there are some exceptions like if h > q then the probability is always 1 and if q > h + n then the probability is always 0.

Using the formula, a cheat probability table for non-ones bid can be created with the rows being the amount of unknown dice or n and the columns being the target faced dice needed or q-h in this case. Each cell probability value is in percent (%).

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	33	0	0	0	0	0	0	0	0	0
2	55	11	0	0	0	0	0	0	0	0
3	70	25	3	0	0	0	0	0	0	0
4	80	40	11	1	0	0	0	0	0	0
5	86	53	20	4	~0	0	0	0	0	0
6	91	64	31	10	1	~0	0	0	0	0
7	94	73	42	17	4	~0	~0	0	0	0
8	96	80	53	25	8	1	~0	~0	0	0
9	97	85	62	34	14	4	~0	~0	~0	0
10	98	89	70	44	21	7	1	~0	~0	~0
11	99	92	76	52	28	12	3	~0	~0	~0
12	99	94	81	60	36	17	6	1	~0	~0
13	99	96	86	67	44	24	10	3	~0	~0
14	99	97	89	73	52	31	14	5	1	~0
15	99	98	92	79	59	38	20	8	3	~0
16	99	98	94	83	66	45	26	12	4	1
17	99	99	95	86	71	52	32	17	7	2
18	99	99	96	89	76	58	39	22	10	4
19	99	99	97	92	81	64	45	27	14	6
20	99	99	98	93	84	70	52	33	19	9
21	99	99	98	95	87	75	58	39	23	12
22	99	99	99	96	90	79	63	45	29	16
23	99	99	99	97	92	83	68	51	34	20
24	99	99	99	98	94	86	73	57	40	25
25	99	99	99	98	95	88	77	62	46	30

Table 1. Non-ones face probability.

As you'd expect, the entirety of the probabilities will not fit into one table sheet. So, there's this program anybody could run using python here:

```
def probability(q: int, h: int, n: int) -> float:
    if h > q:
        return 1
    if q > n + h:
        return 0

    prob = 0
    for i in range(q – h, n + 1):
        prob += C(n, i) * ((2 / 6) ** i) * ((4 / 6) ** (n – i))
return prob
```

As for the probability of one faced bid, can be calculated using similar formula as the non-one faced bid:

$$Probability(q,h) = \sum_{i=q-h}^{n} C(n,i) \cdot \left(\frac{1}{6}\right)^{i} \cdot \left(\frac{5}{6}\right)^{n-i}$$

Also using the formula, with the same format cheat table as the previous one, which the rows represents the amount of unknown dice or n and the columns being the target faced dice needed or q-h. Each cell probability value is in percent (%).

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	16	0	0	0	0	0	0	0	0	0
2	30	2	0	0	0	0	0	0	0	0
3	42	7	~0	0	0	0	0	0	0	0
4	51	13	1	~0	0	0	0	0	0	0
5	59	19	3	~0	~0	0	0	0	0	0
6	66	26	6	~0	~0	~0	0	0	0	0
7	72	33	9	1	~0	~0	~0	0	0	0
8	76	39	13	3	~0	~0	~0	~0	0	0
9	80	45	17	4	~0	~0	~0	~0	~0	0
10	83	51	22	6	1	~0	~0	~0	~0	~0
11	86	56	27	9	2	~0	~0	~0	~0	~0
12	90	61	32	12	3	~0	~0	~0	~0	~0
13	93	66	37	15	5	1	~0	~0	~0	~0
14	94	70	42	19	6	1	~0	~0	~0	~0
15	95	74	46	23	8	2	~0	~0	~0	~0
16	96	77	51	27	11	3	1	~0	~0	~0
17	96	80	55	31	13	5	1	~0	~0	~0
18	97	82	59	35	16	6	2	~0	~0	~0
19	97	84	63	39	19	8	2	~0	~0	~0
20	98	86	67	43	23	10	3	1	~0	~0
21	98	88	70	47	26	12	4	1	~0	~0
22	98	90	73	51	29	14	6	2	~0	~0
23	98	91	76	54	33	17	7	2	~0	~0
24	98	92	78	58	37	19	9	3	1	~0
25	98	93	81	61	40	22	10	3	1	~0

Table 2. Ones face probability.

Additionally, this probability version, could be use in the liar's dice variant which the game doesn't use ones as wild value. Also, there's this function on python for every other probabilities.

```
def probability(q: int, h: int, n: int) -> float:
    if h > q:
        return 1
    if q > n + h:
        return 0

    prob = 0
    for i in range(q - h, n + 1):
        prob += C(n, i) * ((1 / 6) ** i) * ((5 / 6) ** (n - i))
return prob
```

B. Common Hand Version

The other variant is more complex to calculate than the previous one considering there are 9 different hands that is all quite unique to calculate.

1. **Runt**
   Runt is just any combinations of dice so the probability of getting runt is 1 or 100%.

2. **Pair**
   The Probability of getting a pair is 100% minus all different faces dice.
   $$Prob = 1 - \frac{P(6,5)}{6^5} = 90{,}74\%$$
3. **Two Pair**
   There are 6 * 5 = 30 different possible combinations of two pairs. With each two pairs combination has P(5, 2) different positioning and also 1 dice that could be any face.
   $$Prob = \frac{6 \cdot 5 \cdot P(5,2) \cdot 6}{6^5} = 46{,}29\%$$
4. **Low Straight**
   There is only one possible combination of dice which is 1, 2, 3, 4, 5. So, all of the combination possible is 5!.
   $$Prob = \frac{5!}{6^5} = 1{,}54\%$$
5. **Three of a Kind**
   There are 6 possible three of a kind with 2 dice that could be any face. With each kind has P(4, 2) different positioning.
   $$Prob = \frac{6 \cdot 6 \cdot 6 \cdot P(4,2)}{6^5} = 33{,}33\%$$
6. **Full House**
   There are 6 * 5 = 30 possible full house combinations. With each kind has P(4, 2) different positioning.
   $$Prob = \frac{6 \cdot 5 \cdot P(4,2)}{6^5} = 4{,}62\%$$
7. **High Straight**
   Same as low straight but with combination of 2, 3, 4, 5, 6 dice.
   $$Prob = \frac{5!}{6^5} = 1{,}54\%$$
8. **Four of a Kind**
   There are 6 possible four of a kind with each kind has P(5, 2) different position.
   $$Prob = \frac{6 * P(5,2)}{6^5} = 1{,}54\%$$
9. **Five of a Kind**
   There are 6 possible five of a kind with only 1 positioning.
   $$Prob = \frac{6}{6^5} = 0{,}07\%$$

Based on the probabilities above, it can be concluded in the probability table below.

Category	Probability
Five of a Kind	0.07%
Four of a Kind	1.54%
High Straight	1.54%
Full House	4.62%
There of a Kind	33.33%
Low Straight	1.54%
Two Pair	46.29%
Pair	90.74%

Runt	100%

Table 3. Common hand probability.

C. Playtesting

The playtesting is done from the game Liar's Bar using one of the game available in it which is also called by the name, Liar's dice. The rules are similar to what has been explained so far of the original version with the exception of when the player loses in Liar's Bar, they lose a life.

Image 6. Playtesting 1 of Liar's dice.

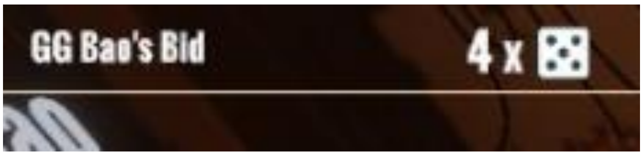

Image 7. The opponent's bid

For this example, there are two players total with each player having 5 dice. we can see that the opponent bids 4 amount of 5 faced dice.

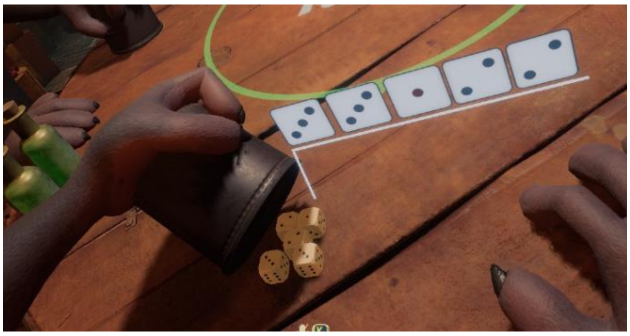
Image 8. The player's revealed dice.

Our player meanwhile only have 1 dice that is relevant to the bid. That means we have a target dice needed of 3 with the amount of unknown dice of 5. If we see at the table, this means that the probability of it being a successful bid is 20% which is quite low. So, it is a good play to call challenge the opponent and call "liar" on them.

Image 9. The player challenges the opponent’s bid.

Image 10. The opponent loses the challenge.

The challenge turns out to be successful with 5 faced dice totaling to 3 instead of 4 making the opponent lose a life.

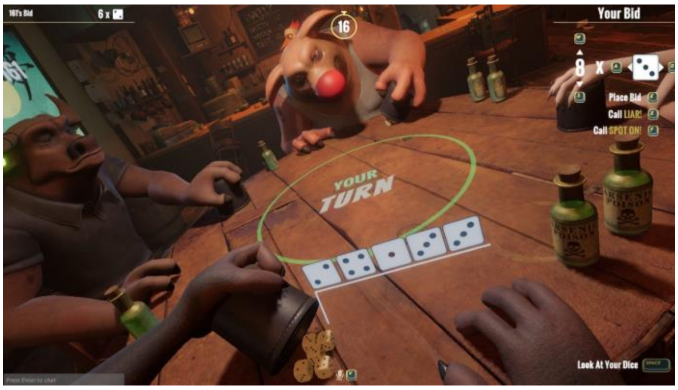

Image 11. Playtesting 2 of Liar’s dice.

For the next example, there are 3 other players resulting in 15 unknown dice.

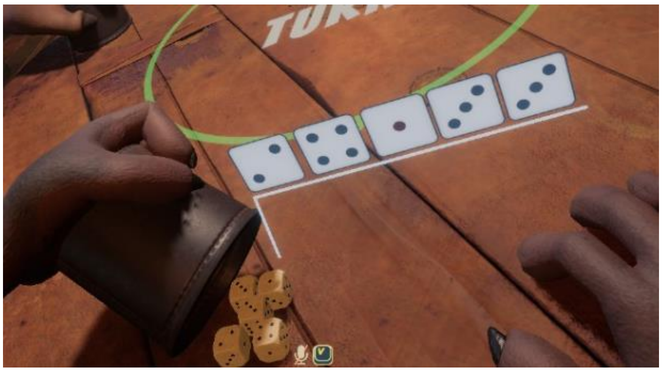

Image 12. The player’s known dice.

As we could see, our dice has 3 amount of 3 face value including the wild 1. If we see at the table for 15 unknown dice, we know that the column five is a safe bid considering the chances of being correct on that bid is 59%. Since we already have 3 amount of 3 face value, we could bid 3 + 5 amount of 3 face value here.

Image 13. The player bids 8 amount of 3 faced dice.

Image 14. The opponent loses a challenge to the player’s bid

The opponent actually challenges our bid here. But, after counting the dice, it turns out there was really 8 amount of 3 faced dice in the pool proving our strategy works!

IV. CONCLUSION

The paper demonstrates the effectiveness of combinatorics in optimizing gameplay strategies in Liar's Dice. By systematically applying permutations, combinations, and probability formulas, we calculate the likelihood of different outcomes, enabling players to make informed bids and challenges. The probability tables for various scenarios in the original version and the detailed analysis of hand probabilities in the common hand version highlight the versatility of combinatorics in enhancing strategic gameplay. This approach offers a robust framework for players seeking to refine their decision-making process in Liar's Dice, emphasizing the power of mathematical reasoning in games of chance and strategy.

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PERNYATAAN

Dengan ini saya menyatakan bahwa makalah yang saya tulis ini adalah tulisan saya sendiri, bukan saduran, atau terjemahan dari makalah orang lain, dan bukan plagiasi.

Bandung, 5 Januari 2024

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